Homework 8

Instructions: The file must be in pdf extension. Show neat and complete work and make sure that your scan is legible. Label your solutions and make sure they are in increasing order.

1 Solving ODEs with Laplace transform

Pierre-Simon ate differential equations and became algebraic.

1. Solve the equation

$$x''(t) - x(t) = \sin(2t), \quad x(0) = 1, \quad x'(0) = 1$$

2. Solve the equation

$$x''(t) + x'(t) = t^2, \quad x(0) = -1, \quad x'(0) = 1$$

3. Solve the equation

$$x''(t) + x(t) = te^{-2t}, \quad x(0) = 0, \quad x'(0) = 1$$

(Hint: Use the shifting property from HW 7 or look at section 3 below.)

2 Heaviside Function

My signals are discontinuous. Oliver to the rescue.

4. Plot the Heaviside function $u(t)$ and the shifted Heaviside function $u(t-5)$.

5. Plot the function $f(t) = u(t-\pi)\sin(t)$. How does it differ from the graph of $\sin(t)$?

6. Rewrite the function

$$f(t) = \begin{cases} 0 & t < 5 \\ t^2 & t \geq 5 \end{cases}$$

using the Heaviside function.

3 Preparation

My Laplace table doesn't have $\mathcal{L}^{-1}\{e^{-s}\frac{1}{s^2}\}$.

Remark. *You proved in class that*

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} \tag{1}$$

This will be useful for two purposes:

- Finding Laplace transform of functions like $\sin(t)u(t-\pi)$. (Note you have $\sin(t)$ and not $\sin(t-\pi)$. But you can find a function such that $f(t-\pi) = \sin(t)$. Look at 7(b).)
- When taking **inverse** Laplace transforms on the right side when your functions look like $e^{-s} \cdot \frac{1}{s^2}$.
  Take inverse Laplace transform on both sides of equation (1):

  $$f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}\mathcal{L}\{f(t)\}\}$$

  Try to write $\frac{1}{s^2}$ as $\mathcal{L}\{\cdot\}$

7. For the following find $f(t)$: (Going from $f(t-a)$ to $f(t)$)
   (a) $f(t-1) = t^3 + 1$
   (b) $f(t-\pi) = \sin(t)$
   (c) $f(t-2) = (t-2)^2 + t$
8. Find the Laplace transforms ($\mathcal{L}$) of the following:
   (a) $\sin(t-\pi)u(t-\pi)$
   (b) $\sin(t)u(t-\pi)$
   (c) $t^2u(t-2)$
   (d) $(t+1)^2u(t-1)$

9. Find the inverse Laplace transforms ($\mathcal{L}^{-1}$) of the following:

   (a) $e^{-s} \cdot \frac{1}{s^2}$

   (b) $e^{-2s} \cdot \frac{s}{s^2-1}$.

   (c) $e^{-3s} \cdot \frac{s}{s^2(s^2+9)}$.

   (d) $e^{-s} \cdot \frac{s}{(s+1)^2+1}$.

4 Finale

Have you heard of the Laplace Transform?

10. Solve the equation

$$x''(t) - x(t) = (t^2 - 1)u(t-1), \quad x(0) = 1, \quad x'(0) = 2$$

11. Define

$$f(t) = \begin{cases} (t-1)^2 & 1 \le t < 2 \\ 3 - t & 2 \le t < 3 \\ 0 \text{ otherwise} & \end{cases}$$

   (a) Sketch the graph of $f(t)$.

   (b) Write down $f(t)$ using the Heaviside funciton.

   (c) Solve $x'' + x = f(t)$, $x(0) = 0, x'(0) = 0$.